

CANONICAL SEMI-RINGS OF FINITE GRAPHS AND TROPICAL CURVES

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ABSTRACT. For a projective curve C and the canonical divisor K_C on C , it is classically known that the canonical ring $R(C) = \bigoplus_{m=0}^{\infty} H^0(C, mK_C)$ is finitely generated in degree at most three. In this article, we study whether analogous statements hold for finite graphs and tropical curves. For any finite graph G , we show that the canonical semi-ring $R(G)$ is finitely generated but that the degree of generators are not bounded by a universal constant. For any hyperelliptic tropical curve Γ with integer edge-length, we show that the canonical semi-ring $R(\Gamma)$ is not finitely generated, and, for tropical curves with integer edge-length in general, we give a sufficient condition for non-finite generation.

1. INTRODUCTION

Let $\mathbb{R}^{\text{trop}} = (\mathbb{R}, \oplus, \odot)$ be the tropical semifield, where the tropical sum \oplus is taking the maximum $a \oplus b := \max\{a, b\}$, and the tropical product \odot is taking the ordinary sum $a \odot b := a + b$. Let $\mathbb{Z}^{\text{trop}} = (\mathbb{Z}, \oplus, \odot)$ be the sub-semifield of \mathbb{R}^{trop} .

A tropical curve Γ is a metric graph with possibly unbounded edges. Equivalently, in a more formal form, a tropical curve is a compact topological space homeomorphic to a one-dimensional simplicial complex equipped with an integral affine structure over $\mathbb{R}^{\text{trop}} \cup \{-\infty\}$ (see [MZ]). Finite graphs are seen as a discrete version of tropical curves.

In relation to the classical algebraic curves, tropical curves and finite graphs have been much studied recently. For example, the Riemann-Roch formula on finite graphs and tropical curves (analogous to the classical Riemann-Roch formula on algebraic curves) are established in [BN, GK, MZ]. The Clifford theorem is established in [Co, Fa].

In this article, we consider whether the analogy of the following classical theorem holds or not.

Theorem 1.1 (Riemann, Max Noether). *Let C be a smooth complex projective curve of genus $g \geq 2$, and let K_C be the canonical divisor on C . Let $R(C) := \bigoplus_{m=0}^{\infty} H^0(C, mK_C)$ be the canonical ring. Then:*

- (a) *$R(C)$ is finitely generated as a graded ring over \mathbb{C} .*
- (b) *$R(C)$ is generated in degree at most three.*

Our first result is that for a finite graph G , the analogous statement (a) holds, but that the degrees of generators cannot be bounded by a universal constant. For a divisor D on G , let $R(G, D)$ be the set of rational functions f on G such that $D + \text{div}(f)$ is effective (see [BN] for details). We also refer to §3.1 for terminology.

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We show that the direct sum $\bigoplus_{m=0}^{\infty} R(G, mD)$ has a graded semi-ring structure over \mathbb{Z}^{trop} for any finite graph G and any divisor D on G (Lemma 3.5). Then the following is the first result:

Theorem 1.2 (Theorem 3.6, Theorem 3.7). *Let G be a finite graph and let K_G be the canonical divisor on G . We set $R(G) := \bigoplus_{m=0}^{\infty} R(G, mK_G)$. Then:*

- (a) *$R(G)$ is finitely generated as a graded semi-ring over \mathbb{Z}^{trop} .*
- (b) *For any integer $n \geq 1$, there exists a finite graph G_n such that $R(G_n)$ is not generated in degree at most $n - 1$.*

For (a), we show that, in fact, the semi-ring $\bigoplus_{m=0}^{\infty} R(G, mD)$ is finitely generated as a graded semi-ring over \mathbb{Z}^{trop} for any divisor D on G .

Our next result is that for a tropical curve Γ with integer edge-length, the analogous statement (a) does *not* hold in general (hence neither (b)). We give a sufficient condition for non-finite generation of the canonical semi-ring of tropical curves. For a divisor D on Γ , let $R(\Gamma, D)$ be the set of rational functions f on Γ such that $D + \text{div}(f)$ is effective (see [HMY] for details). We also refer to §2.1 for terminology. We show that the direct sum $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ has a graded semi-ring structure over \mathbb{R}^{trop} for any tropical curve Γ and any divisor D on Γ (Lemma 2.5). Then the following is the second result:

Theorem 1.3 (Corollary 2.12). *Let Γ be a \mathbb{Z} -tropical curve of genus $g \geq 2$, and let K_{Γ} be the canonical divisor on Γ . Assume that there exist an edge e of the canonical model of Γ and a positive integer n such that e is not a bridge and nK_{Γ} is linearly equivalent to $n(g-1)[p] + n(g-1)[q]$, where p and q are the endpoints of e . Then the canonical semi-ring $R(\Gamma) := \bigoplus_{m=0}^{\infty} R(\Gamma, mK_{\Gamma})$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} .*

Corollary 1.4 (Corollary 2.13). (a) *Let Γ be a hyperelliptic \mathbb{Z} -tropical curve of genus at least 2. Then $R(\Gamma)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} .*
 (b) *Let K be a complete graph on vertices at least 4, and let Γ be the tropical curve associated to K , where each edge of K is assigned the same positive integer as length. Then $R(\Gamma)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} .*

For Theorem 1.3, we give, in fact, a sufficient condition for non-finite generation of the graded semi-ring $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ over \mathbb{R}^{trop} for any \mathbb{Z} -divisor D of degree at least 2 on a \mathbb{Z} -tropical curve Γ (Theorem 2.7).

It seems likely that, for *any* tropical curve of genus $g \geq 2$, the canonical semi-ring $R(\Gamma) = \bigoplus_{m=0}^{\infty} R(\Gamma, mK_{\Gamma})$ will not be finitely generated as a graded semi-ring over \mathbb{R}^{trop} , which we pose as a question.

For the proof of Theorem 1.3, we use the notion of *extremals* of $R(\Gamma, D)$ introduced by Haase, Musiker and Yu [HMY]. Then Theorem 1.2(b) is deduced as a certain discrete version of Theorem 1.3. Theorem 1.2(a) is shown by using Gordan's lemma (see [Fu, p.12, Proposition 1]).

2. TROPICAL CURVES

In this section, we prove Theorem 1.3 and Corollary 1.4.

2.1. Theory of divisors on tropical curves. In this section, we first put together necessary definitions and results on the theory of divisors on tropical curves, which will be used later. Our basic references are [Ch, GK, HMY].

In this article, all finite graphs are assumed to be connected and allowed to have loops and multiple edges. For a finite graph G , let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges, respectively. A *bridge* is an edge of G which makes G disconnected.

A metric space Γ is called a *metric graph* if there exist a finite graph G and a function $l : E(G) \rightarrow \mathbb{R}_{>0}$ (called the edge-length function) such that Γ is obtained by gluing the intervals $[0, l(e)]$ for $e \in E(G)$ at their endpoints so as to keep the combinatorial data of G . The pair (G, l) is called a *model* for Γ .

In this article, we assume that a metric space Γ is not homeomorphic to the circle S^1 . For a point x of Γ , we define the *valence* $\text{val}(x)$ of x to be the number of connected components in $U_x \setminus \{x\}$ for any sufficiently small neighborhood U_x of x .

Let V be a finite subset of Γ which includes all points of valence different from 2, and G_V be a finite graph whose vertices are the points in V and whose edges correspond to the connected components of $\Gamma \setminus V$. If we define a function $l : E(G_V) \rightarrow \mathbb{R}_{>0}$ such that $l(e)$ is equal to the length of the corresponding component for each edge e , then (G_V, l) is a model for Γ . The model (G_V, l) is called the *canonical model* for Γ if we take the set of all points of valence different from 2 as the finite subset V .

A metric graph Γ with the canonical model (G, l) is called a \mathbb{Z} -*metric graph* if $l(e)$ is an integer for each edge e of G . In this case, the points of Γ with integer distance to the vertices of G are called \mathbb{Z} -*points*, and we denote the set of \mathbb{Z} -points by $\Gamma_{\mathbb{Z}}$.

Tropical curves are defined in a similar way as metric graphs. A metric space Γ is called a *tropical curve* if there exist a finite graph G and a function $l : E(G) \rightarrow \mathbb{R}_{>0} \cup \{\infty\}$ such that Γ is obtained by gluing the intervals $[0, l(e)]$ for $e \in E(G)$ at their endpoints so as to keep the combinatorial data of G , where the only edges adjacent to a one-valent vertex may have length ∞ . The pair (G, l) is called a *model* for Γ .

We define the *canonical model* of a tropical curve in the same way as that of a metric graph. A tropical curve Γ with the canonical model (G, l) is called a \mathbb{Z} -*tropical curve* if $l(e)$ is either an integer or equal to ∞ for each edge e of G . In this case, the points of Γ with integer distance to the vertices of G are called \mathbb{Z} -*points*.

A *divisor* on a tropical curve Γ is a finite formal sum of points of Γ , and a \mathbb{Z} -*divisor* on a \mathbb{Z} -tropical curve Γ is a finite formal sum of \mathbb{Z} -points of Γ . We denote the set of all divisors on Γ by $\text{Div}(\Gamma)$. If D is a divisor on Γ , we write it as $D = \sum_{x \in \Gamma} D(x)[x]$, where $D(x)$ is an integer and $[x]$ is merely a symbol. For a divisor D , we define the *degree* $\deg(D)$ to be the integer $\deg(D) := \sum_{x \in \Gamma} D(x)$, and the *support* $\text{Supp}(D)$ to be the set of all points of Γ occurring in D with a non-zero coefficient. A divisor D is called *effective*, and we write $D \geq 0$, if $D(x)$ is a non-negative integer for all $x \in \Gamma$. On a \mathbb{Z} -tropical curve, a divisor D is called a \mathbb{Z} -*divisor* if $\text{Supp}(D)$ is a subset of $\Gamma_{\mathbb{Z}}$. The *canonical divisor* on a tropical curve Γ is defined to be $K_{\Gamma} := \sum_{x \in \Gamma} (\text{val}(x) - 2)[x]$.

A *rational function* f on a tropical curve Γ is a continuous function $f : \Gamma \rightarrow \mathbb{R} \cup \{\pm\infty\}$ that is piecewise linear with finitely many pieces and integer slopes, and may take on values $\pm\infty$ only at the one-valent points. The set of all rational

functions on Γ is denoted by $\text{Rat}(\Gamma)$. For a rational function f and a vertex x , we define the *order* $\text{ord}_x(f)$ of f at x as the sum of outgoing slopes at x . The *principal divisor* associated to f is defined to be

$$\text{div}(f) := \sum_{x \in \Gamma} \text{ord}_x(f)[x].$$

We say that two divisors D and D' are *linearly equivalent*, and we write $D \sim D'$, if there exists a rational function f such that $D - D' = \text{div}(f)$.

Now, we define the most important objects in this article.

Definition 2.1. Let D be a divisor on a tropical curve Γ . We set

$$R(\Gamma, D) := \{f \in \text{Rat}(\Gamma) \mid \text{div}(f) + D \geq 0\}.$$

For $f, g \in R(\Gamma, D)$ and $c \in \mathbb{R}^{\text{trop}}$, we define the tropical sum $f \oplus g$ and the tropical \mathbb{R}^{trop} -action $c \odot f$ as follows:

$$\begin{aligned} (f \oplus g)(x) &:= \max\{f(x), g(x)\}, \\ (c \odot f)(x) &:= c + f(x). \end{aligned}$$

An *extremal* of $R(\Gamma, D)$ is an element such that $f = g \oplus h$ implies $f = g$ or $f = h$ for any $g, h \in R(\Gamma, D)$. A subset $\Gamma' \subset \Gamma$ is called a *subgraph* if Γ' is a compact subset with a finite number of connected components. For a subgraph Γ' and a positive real number l , we define the rational function *chip firing move* $\text{CF}(\Gamma', l)$ as

$$\text{CF}(\Gamma', l)(x) := -\min\{l, \text{dist}(x, \Gamma')\}.$$

We say that a subgraph Γ' can *fire* on a divisor D if the divisor $D + \text{div}(\text{CF}(\Gamma', l))$ is effective for a sufficiently small positive real number l . Here, by a sufficiently small positive real number, we mean that l is chosen to be small enough so that the “chips” do not pass through each other or pass through points of valence 2.

Proposition 2.2 ([HMY, Lemma 4, Theorem 6, Corollary 9]). (a) $R(\Gamma, D)$ is a semi-module over \mathbb{R}^{trop} .

- (b) The set of extremals of $R(\Gamma, D)$ is finite modulo \mathbb{R}^{trop} -action.
- (c) $R(\Gamma, D)$ is generated by the extremals.

The following lemma is useful for finding extremals:

Lemma 2.3 ([HMY, Lemma 5]). A rational function f is an extremal of $R(\Gamma, D)$ if and only if there are not two proper subgraphs Γ_1 and Γ_2 covering Γ such that each can fire on $D + \text{div}(f)$.

2.2. Proofs of Theorem 1.3 and Corollary 1.4.

Definition 2.4 (Canonical semi-rings of tropical curves). Let Γ be a tropical curve, and let K_Γ be the canonical divisor. The direct sum $\bigoplus_{m=0}^{\infty} R(\Gamma, mK_\Gamma)$ is called the *canonical semi-ring* of Γ , and denoted by $R(\Gamma)$.

For $f \in R(\Gamma, nD)$ and $g \in R(\Gamma, kD)$, we define the tropical product $f \odot g$ as

$$(f \odot g)(x) := f(x) + g(x).$$

We show that $R(\Gamma)$ has indeed a graded semi-ring structure over \mathbb{R}^{trop} .

Lemma 2.5. Let Γ be a tropical curve. Then the canonical semi-ring $R(\Gamma)$ has naturally a graded semi-ring structure over \mathbb{R}^{trop} . For any divisor D on Γ , in general, the direct sum $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ has naturally a graded semi-ring structure over \mathbb{R}^{trop} .

Proof. We prove only the general case. Let f and g be elements of $R(\Gamma, nD)$ and $R(\Gamma, kD)$, respectively. Since the order of a rational function at a point is defined as the sum of outgoing slopes and the tropical product is defined as the ordinary sum, it follows that $\text{div}(f \odot g) = \text{div}(f) + \text{div}(g)$. Therefore $(n+k)D + \text{div}(f \odot g) = nD + \text{div}(f) + kD + \text{div}(g)$. Here both $nD + \text{div}(f)$ and $kD + \text{div}(g)$ are effective, so $(n+k)D + \text{div}(f \odot g)$ is also effective. This means that the tropical product $f \odot g$ is an element of $R(\Gamma, (n+k)D)$. Together Proposition 2.2(a), we obtain the assertion. \square

Remark 2.6. Since we have $R(\Gamma, 0D) = \mathbb{R}^{\text{trop}}$, the semi-ring $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ can be seen as a semi-ring over the 0-th part $R(\Gamma, 0D)$.

Theorem 2.7 (Sufficient condition for non-finite generation). *Let Γ be a \mathbb{Z} -tropical curve of genus $g \geq 2$, and let D be a \mathbb{Z} -divisor of degree $d \geq 2$. Assume that there exist an edge e of the canonical model of Γ and a positive integer n such that e is not a bridge and nD is linearly equivalent to $\frac{nd}{2}[p] + \frac{nd}{2}[q]$, where p and q are the endpoints of e . Then $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} .*

Proof. Let L be the length of e . Note that, if e is a loop, then $p = q$. We begin by showing the following lemma.

Lemma 2.8. *Let D , p , q be as in Theorem 2.7. If there exists a positive integer s such that sD is linearly equivalent to $\frac{sd}{2}[p] + \frac{sd}{2}[q]$, then there exists an extremal of $R(\Gamma, 2sLD)$ which is not generated by elements of $\bigoplus_{m=0}^{2sL-1} R(\Gamma, mD)$ over \mathbb{R}^{trop} .*

Proof. Put $N := sd$. Since sD is linearly equivalent to $\frac{N}{2}[p] + \frac{N}{2}[q]$, it follows that $2sLD$ is linearly equivalent to $LN[p] + LN[q]$. Identify the edge e with an interval $[0, L]$ such that p and q are identified with 0 and L , respectively. Let r be the point identified with the point $\frac{LN}{2LN-1}L$ of the interval. By definition, r is not a \mathbb{Z} -point.

First, we show two claims.

Claim 2.9. *The divisor $2sLD$ is linearly equivalent to $[p] + (2LN - 1)[r]$.*

Proof. Let \tilde{f} be the rational function which takes on value 0 on $\Gamma \setminus e$, and value $-\frac{LN(LN-1)}{2LN-1}L$ at r , and is extended linearly to $e \setminus \{r\}$. Then the orders of \tilde{f} at p , q , and r are

$$\text{ord}_p(\tilde{f}) = -(LN - 1),$$

$$\text{ord}_q(\tilde{f}) = -LN,$$

$$\text{ord}_r(\tilde{f}) = 2LN - 1.$$

Moreover, the order of \tilde{f} at any point of $\Gamma \setminus \{p, q, r\}$ is equal to 0 by construction. From these values we conclude that $LN[p] + LN[q] + \text{div}(\tilde{f})$ is equal to $[p] + (2LN - 1)[r]$. Therefore $2sLD$ is linearly equivalent to $[p] + (2LN - 1)[r]$. \square

Claim 2.10. *Let f be the rational function such that $2sLD + \text{div}(f) = [p] + (2LN - 1)[r]$. Then f is an extremal of $R(\Gamma, 2sLD)$.*

Proof. Since p is an endpoint of e and e is an edge of the canonical model, we have $\text{val}(p) \neq 2$. Moreover, by the assumption that e is not a bridge, we have

$\text{val}(p) \geq 3$. Suppose that Γ_1 is a subgraph of Γ that can fire on $2sLD + \text{div}(f) = [p] + (2LN - 1)[r]$. Then the boundary set $\partial\Gamma_1$ of Γ_1 in Γ is contained in $\{p, r\}$. Since $\text{val}(p) \geq 3$, we have $\Gamma_1 = \Gamma \setminus (p, r)$ or $\Gamma_1 = \{r\}$. (Here (p, r) denotes the open interval in e connecting p to r .) From Lemma 2.3 we conclude that f is an extremal of $R(\Gamma, 2sLD)$. \square

We prove that f is not generated by elements of $\bigoplus_{m=0}^{2sL-1} R(\Gamma, mD)$ over \mathbb{R}^{trop} by contradiction. Suppose that f is generated by elements of $\bigoplus_{m=0}^{2sL-1} R(\Gamma, mD)$ over \mathbb{R}^{trop} . Then we have

$$f = \sum_{I=\{i_1 \leq \dots \leq i_{|I|}\} \subset \{0, 1, \dots, 2sL-1\}} f_{i_1} \odot \dots \odot f_{i_{|I|}},$$

where f_i is an element of $R(\Gamma, iD)$ and the sum $\sum_{i \in I} i$ is equal to $2sL$ for each I . Note that there are at least two terms in $f_{i_1} \odot \dots \odot f_{i_{|I|}}$ for each I . By Lemma 2.5 we can take $1 \leq l_I, k_I \leq 2sL - 1$ and $g_I \in R(\Gamma, l_I D)$ and $h_I \in R(\Gamma, k_I D)$ such that $f_{i_1} \odot \dots \odot f_{i_{|I|}} = g_I \odot h_I$ and $l_I + k_I = 2sL$. By Proposition 2.2(b) we may assume that g_j and h_j are the extremals of $R(\Gamma, l_j D)$ and $R(\Gamma, k_j D)$, respectively. Then we have

$$f = (g_1 \odot h_1) \oplus \dots \oplus (g_\alpha \odot h_\alpha),$$

where each g_j and h_j is an extremal of $R(\Gamma, l_j D)$ and $R(\Gamma, k_j D)$, respectively. Since f is an extremal, it follows that f is equal to $g_1 \odot h_1$ after changing indices if necessary.

Put $g := g_1$, $h := h_1$, $l := l_1$, and $k := k_1$. Recall that $1 \leq l, k \leq 2sL - 1$. Now, we have

$$[p] + (2LN - 1)[r] = 2sLD + \text{div}(f) = lD + \text{div}(g) + kD + \text{div}(h).$$

Since both $lD + \text{div}(g)$ and $kD + \text{div}(h)$ are effective, we may assume that

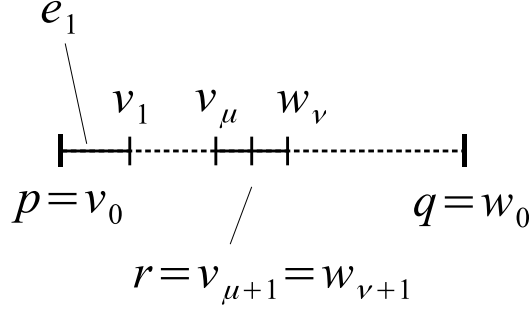
$$\begin{aligned} lD + \text{div}(g) &= [p] + (ld - 1)[r], \\ kD + \text{div}(h) &= kd[r], \end{aligned}$$

after changing the role of g and h if necessary.

In this setting, we deduce a contradiction by studying the property of the rational function h . Since $\text{div}(h) = kd[r] - kD$, all the zeros and poles of h lie in $\text{Supp}(D) \cup \{r\}$. Let $v_0, \dots, v_\mu, r, w_\nu, \dots, w_0$ be the points of $e \cap (\text{Supp}(D) \cup \{p\} \cup \{q\} \cup \{r\})$ in this order, where v_0 is p and w_0 is q . Moreover, let $e_i (i = 1, \dots, \mu + 1)$ be the segment which connects v_{i-1} to v_i , and $\tilde{e}_j (j = 1, \dots, \nu + 1)$ be the segment which connects w_{j-1} to w_j , where we set $v_{\mu+1} = r$ and $w_{\nu+1} = r$. We denote the each length of e_i and \tilde{e}_j by $l(e_i)$ and $l(\tilde{e}_j)$, respectively. The sum of outgoing slopes of h at x as a rational function on e_i and \tilde{e}_j are denoted by $\text{ord}_x(h|_{e_i})$ and $\text{ord}_x(h|_{\tilde{e}_j})$, respectively.

Since h is continuous, we have

$$\begin{aligned} h(r) &= h(p) + \sum_{i=1}^{\mu+1} \text{ord}_{v_{i-1}}(h|_{e_i}) l(e_i) \\ &= h(q) + \sum_{j=1}^{\nu+1} \text{ord}_{w_{j-1}}(h|_{\tilde{e}_j}) l(\tilde{e}_j) \end{aligned}$$

FIGURE 1. On the edge e

Now, by the equality $\text{div}(h) = kd[r] - kD$, we have

$$\begin{aligned} \text{ord}_{v_i}(h|_{e_i}) + \text{ord}_{v_i}(h|_{e_{i+1}}) &= \text{ord}_{v_i}(h) = -kD(v_i), \\ \text{ord}_{w_j}(h|_{\tilde{e}_j}) + \text{ord}_{w_j}(h|_{\tilde{e}_{j+1}}) &= \text{ord}_{w_j}(h) = -kD(w_j), \end{aligned}$$

for $i = 1, \dots, \mu$, and for $j = 1, \dots, \nu$, and we have

$$\begin{aligned} \text{ord}_{v_i}(h|_{e_{i+1}}) &= -\text{ord}_{v_{i+1}}(h|_{e_{i+1}}), \\ \text{ord}_{w_j}(h|_{\tilde{e}_{j+1}}) &= -\text{ord}_{w_{j+1}}(h|_{\tilde{e}_{j+1}}), \end{aligned}$$

for $i = 0, \dots, \mu$, and for $j = 0, \dots, \nu$.

From these relations, we deduce that

$$\begin{aligned} \text{ord}_{v_i}(h|_{e_{i+1}}) &= -\text{ord}_{v_{i+1}}(h|_{e_{i+1}}) \\ &= kD(v_{i+1}) + \text{ord}_{v_{i+1}}(h|_{e_{i+2}}) \\ &= kD(v_{i+1}) - \text{ord}_{v_{i+2}}(h|_{e_{i+2}}) \\ &= \dots \\ (2.1) \quad &= k \sum_{\alpha=i+1}^{\mu} D(v_{\alpha}) + \text{ord}_{v_{\mu}}(h|_{e_{\mu+1}}) \\ &= k \sum_{\alpha=i+1}^{\mu} D(v_{\alpha}) - \text{ord}_{v_{\mu+1}}(h|_{e_{\mu+1}}). \end{aligned}$$

Similarly, we deduce that

$$(2.2) \quad \text{ord}_{w_j}(h|_{\tilde{e}_{j+1}}) = k \sum_{\beta=j+1}^{\nu} D(w_{\beta}) - \text{ord}_{w_{\nu+1}}(h|_{\tilde{e}_{\nu+1}}).$$

Since D is a \mathbb{Z} -divisor and r is not a \mathbb{Z} -point, we have $r \notin \text{Supp}(D)$. Thus

$$(2.3) \quad \text{ord}_{v_{\mu+1}}(h|_{e_{\mu+1}}) + \text{ord}_{w_{\nu+1}}(h|_{\tilde{e}_{\nu+1}}) = \text{ord}_r(h) = kd.$$

It follows that

$$h(p) - h(q)$$

$$\begin{aligned}
&= \sum_{j=1}^{\nu+1} \text{ord}_{w_{j-1}}(h|_{\tilde{e}_j}) l(\tilde{e}_j) - \sum_{i=1}^{\mu+1} \text{ord}_{v_{i-1}}(h|_{e_i}) l(e_i) \\
&= \sum_{j=1}^{\nu} \left(k \sum_{\beta=j}^{\nu} D(w_{\beta}) - \text{ord}_{w_{\nu+1}}(h|_{\tilde{e}_{\nu+1}}) \right) l(\tilde{e}_j) - \text{ord}_{w_{\nu+1}}(h|_{\tilde{e}_{\nu+1}}) l(\tilde{e}_{\nu+1}) \\
&\quad - \sum_{i=1}^{\mu} \left(k \sum_{\alpha=i}^{\mu} D(v_{\alpha}) - \text{ord}_{v_{\mu+1}}(h|_{e_{\mu+1}}) \right) l(e_i) + \text{ord}_{v_{\mu+1}}(h|_{e_{\mu+1}}) l(e_{\mu+1}) \\
&= k \left(\sum_{j=1}^{\nu} l(\tilde{e}_j) \sum_{\beta=j}^{\nu} D(w_{\beta}) - \sum_{i=1}^{\mu} l(e_i) \sum_{\alpha=i}^{\mu} D(v_{\alpha}) \right) \\
&\quad - \text{ord}_{w_{\nu+1}}(h|_{\tilde{e}_{\nu+1}}) \sum_{j=1}^{\nu+1} l(\tilde{e}_j) + \text{ord}_{v_{\mu+1}}(h|_{e_{\mu+1}}) \sum_{i=1}^{\mu+1} l(e_i) \\
&= k \left(\sum_{j=1}^{\nu} l(\tilde{e}_j) \sum_{\beta=j}^{\nu} D(w_{\beta}) - \sum_{i=1}^{\mu} l(e_i) \sum_{\alpha=i}^{\mu} D(v_{\alpha}) \right) \\
&\quad - \text{ord}_{w_{\nu+1}}(h|_{\tilde{e}_{\nu+1}}) \left(\sum_{j=1}^{\nu+1} l(\tilde{e}_j) + \sum_{i=1}^{\mu+1} l(e_i) \right) + kd \sum_{i=1}^{\mu+1} l(e_i),
\end{aligned}$$

where we use (2.1), (2.2) in the second equality, and (2.3) in the last equality.

By construction, we have

$$\begin{aligned}
\sum_{j=1}^{\nu+1} l(\tilde{e}_j) + \sum_{i=1}^{\mu+1} l(e_i) &= l(\tilde{e}_1) + \cdots + l(\tilde{e}_{\nu+1}) + l(e_1) + \cdots + l(e_{\mu+1}) = L, \\
\sum_{i=1}^{\mu+1} l(e_i) &= l(e_1) + \cdots + l(e_{\mu+1}) = \frac{LN}{2LN-1} L.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{kdL^2N}{2LN-1} \\
&= h(p) - h(q) + \text{ord}_r(h|_{\tilde{e}_{\nu+1}})L - k \left(\sum_{j=1}^{\nu} l(\tilde{e}_j) \sum_{\beta=j}^{\nu} D(w_{\beta}) - \sum_{i=1}^{\mu} l(e_i) \sum_{\alpha=i}^{\mu} D(v_{\alpha}) \right).
\end{aligned}$$

Claim 2.11. *The value $\frac{kdL^2N}{2LN-1}$ is an integer.*

Proof. First, we claim that the value $h(p) - h(q)$ is an integer. Indeed, since e is not a bridge, there exists a path γ in $\Gamma \setminus e$ such that p and q are the endpoints of γ . Since h is a piecewise linear function with integer slopes along γ and both zeros and poles of h on $\Gamma \setminus e$ are \mathbb{Z} -points, it follows that the difference $h(p) - h(q)$ is an integer.

Since Γ is a \mathbb{Z} -tropical curve and D is a \mathbb{Z} -divisor, it follows that $l(e_i)$, $l(\tilde{e}_j)$, $D(v_i)$, $D(w_j)$ ($i = 1, \dots, \mu$; $j = 1, \dots, \nu$), L are integers. Moreover, by definition, $\text{ord}_r(h|_{\tilde{e}_{\mu+1}})$ is an integer. Thus $\frac{kdL^2N}{2LN-1}$ is an integer. \square

Since LN and $2LN - 1$, and Ld and $2LN - 1 = 2sLd - 1$ are relatively prime, respectively, it follows that there exists a positive integer M such that $k = M(2LN - 1)$. Then we have

$$k \geq 2LN - 1 = 2sLd - 1 \geq 2sL - 1.$$

Since $k \leq 2sL - 1$, it follows that $k = 2sL - 1$ and $d = 1$, but this contradicts $d \geq 2$.

Thus f is not generated by elements of $\bigoplus_{m=0}^{2sL-1} R(\Gamma, mD)$ over \mathbb{R}^{trop} . \square

Now, we return to the proof of Theorem 2.7. We prove that $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} by contradiction. Suppose that $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ is finitely generated as a graded semi-ring over \mathbb{R}^{trop} .

Since elements of $R(\Gamma, mD)$ is generated by the extremals over \mathbb{R}^{trop} , we may assume that all the generators of $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ is an extremal of $R(\Gamma, mD)$ for some positive integer m . Let M be the maximal number among such numbers. Fix a positive integer k such that kn is bigger than M , and put $s := kn$. Then we have

$$sD = knD \sim \frac{knd}{2}[p] + \frac{knd}{2}[q] = \frac{sd}{2}[p] + \frac{sd}{2}[q].$$

Applying Lemma 2.8, we get an extremal of $R(\Gamma, 2sLD)$ which is not generated by the elements of $\bigoplus_{m=0}^{2sL-1} R(\Gamma, mD)$ over \mathbb{R}^{trop} , but this contradicts the maximality of M . Thus $\bigoplus_{m=0}^{\infty} R(\Gamma, mD)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} . \square

Corollary 2.12 (Sufficient condition for non-finite generation of canonical semi-rings, Theorem 1.3). *Let Γ be a \mathbb{Z} -tropical curve of genus $g \geq 2$. Assume that there exist an edge e of the canonical model of Γ and a positive integer n such that e is not a bridge and nK_{Γ} is linearly equivalent to $n(g-1)[p] + n(g-1)[q]$, where p and q are the endpoints of e . Then the canonical semi-ring $R(\Gamma)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} .*

A finite graph is called a *complete graph* on n vertices if it is a finite graph with n vertices in which every pair of distinct vertices is connected by a unique edge.

Corollary 2.13 (Corollary 1.4). (a) *Let Γ be a hyperelliptic \mathbb{Z} -tropical curve of genus at least 2. Then $R(\Gamma)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} .*

(b) *Let n be an integer at least 4, let K be a complete graph on n vertices, and let Γ be the tropical curve associated to K , where each edge of K is assigned the same positive integer as length. Then $R(\Gamma)$ is not finitely generated as a graded semi-ring over \mathbb{R}^{trop} .*

Proof. For (a), let (G, l) be the canonical model of Γ . By Chan's theorem [Ch, Theorem 3.12], there exists an edge e of G such that $r_{\Gamma}([p] + [q]) = 1$, where p and q are the endpoints of e . By the Riemann-Roch formula, it follows that $(g-1)[p] + (g-1)[q]$ is linearly equivalent to K_{Γ} . Applying Theorem 2.7 (with $n = 1$), the statement (a) follows.

For (b), there are two cases, one is that n is an odd number, and the other is that n is an even number. Fix any two vertices v and w of K , and let e be the unique edge which connects these vertices. Since the genus of Γ is equal to $\frac{n(n-3)}{2} + 1$, the degree of the canonical divisor is equal to $n(n-3)$. If n is odd, then K_{Γ} is equivalent to $\frac{n(n-3)}{2}[v] + \frac{n(n-3)}{2}[w]$. If n is even, then $2K_{\Gamma}$ is equivalent to

$n(n-3)[v] + n(n-3)[w]$. Therefore, in both cases, we can apply Theorem 2.7, and the statement (b) follows. \square

3. FINITE GRAPHS

In this section, we prove Theorem 1.2.

3.1. Theory of divisors on finite graphs. In a similar way as in §2.1, we can establish the theory of divisors on finite graphs. Our basic reference is [BN].

We denote the valence of a vertex x by $\text{val}(x)$. A *divisor* on a finite graph G is a finite formal sum of vertices and we denote the set of all divisors on G by $\text{Div}(G)$. If D is a divisor on G , we write it as $D = \sum_{x \in V(G)} D(x)[x]$, where $D(x)$ is an integer and $[x]$ is merely a symbol. The *degree* of a divisor and an *effective* divisor are defined in the same way as in §2.1. The *canonical divisor* on a finite graph G is defined to be $K_G := \sum_{x \in V(G)} (\text{val}(x) - 2)[x]$.

A *rational function* f on a finite graph G is a \mathbb{Z} -valued function on vertices $V(G)$. The set of all rational functions on G is denoted by $\text{Rat}(G)$. For a rational function f and a vertex x , we define the *order* $\text{ord}_x(f)$ of f at x as the sum of differences between the value at x and at each vertex adjacent to x , that is, we define it to be the integer

$$\text{ord}_x(f) := \sum_{e = \overline{xy} \in E(G)} (f(y) - f(x)),$$

where $e = \overline{xy}$ means that x and y are the endpoints of $e \in E(G)$. The *principal divisor* associated to f is defined to be

$$\text{div}(f) := \sum_{x \in V(G)} \text{ord}_x(f)[x].$$

Definition 3.1. Let D be a divisor on a finite graph G . We set

$$R(G, D) := \{f \in \text{Rat}(G) \mid \text{div}(f) + D \geq 0\}.$$

For $f, g \in R(G, D)$ and $c \in \mathbb{Z}^{\text{trop}}$, we define the tropical sum $f \oplus g$ and the tropical \mathbb{Z}^{trop} -action $c \odot f$ as follows:

$$\begin{aligned} (f \oplus g)(x) &:= \max\{f(x), g(x)\}, \\ (c \odot f)(x) &:= c + f(x). \end{aligned}$$

An *extremal* of $R(G, D)$ is an element such that $f = g \oplus h$ implies $f = g$ or $f = h$ for any $g, h \in R(G, D)$. For a subset V' of vertices $V(G)$, we define the rational function $\text{CF}(V')$ on G as

$$\text{CF}(V')(x) := \begin{cases} 0 & (x \in V'), \\ -1 & (x \notin V'). \end{cases}$$

We say that a subset V' of vertices $V(G)$ can *fire* on a divisor D if the divisor $D + \text{div}(\text{CF}(V'))$ is effective.

Proposition 3.2. (a) $R(G, D)$ is a semi-module over \mathbb{Z}^{trop} .
 (b) The set of extremals of $R(G, D)$ is finite modulo \mathbb{Z}^{trop} -action.
 (c) $R(G, D)$ is generated by the extremals.

Proof. For (a), it is clear that $c \odot f$ is an element of $R(G, D)$ for $c \in \mathbb{Z}^{\text{trop}}$ and $f \in R(G, D)$. So it is sufficient to show that $f \oplus g$ is an element of $R(G, D)$ for any $f, g \in R(G, D)$. If we have $(f \oplus g)(x) = f(x)$ for a fixed vertex x , then it follows that

$$\begin{aligned} \text{ord}_x(f \oplus g) &= \sum_{e=\overline{xy} \in E(G)} ((f \oplus g)(y) - (f \oplus g)(x)) \\ &\geq \sum_{e=\overline{xy} \in E(G)} (f(y) - f(x)) = \text{ord}_x(f). \end{aligned}$$

Similarly, we have $\text{ord}_x(f \oplus g) \geq \text{ord}_x(g)$ if $(f \oplus g)(x) = g(x)$ at x . Since both $D + \text{div}(f)$ and $D + \text{div}(g)$ are effective, it follows that $D + \text{div}(f \oplus g) = D + \sum_{x \in V(G)} \text{ord}_x(f \oplus g)$ is effective. Then the statement (a) follows.

For (b), we identify $R(G, D)$ with the lattice points of a polyhedron in an Euclidean space, which is a finite set. Since the way of identification will be described in the proof of Theorem 3.7, we omit the detail now. Then the statement (b) follows.

The statement (c) is proved in a similar way as [HMY, Corollary 9]. \square

The next lemma is proven in a similar way as [HMY, Lemma 5].

Lemma 3.3. *A rational function f is an extremal of $R(G, D)$ if and only if there are not two proper subsets V_1 and V_2 covering $V(G)$ such that each can fire on $D + \text{div}(f)$.*

Definition 3.4 (Canonical semi-rings of finite graphs). Let G be a finite graph, and let K_G be the canonical divisor. The direct sum $\bigoplus_{m=0}^{\infty} R(G, mK_G)$ is called the *canonical semi-ring* of G , and denoted by $R(G)$.

For $f \in R(G, nD)$ and $g \in R(G, kD)$, we define the tropical product $f \odot g$ as

$$(f \odot g)(x) := f(x) + g(x).$$

In the same way as Lemma 2.5, we can prove the next lemma, that is, that $R(G)$ has indeed a graded semi-ring structure over \mathbb{Z}^{trop} .

Lemma 3.5. *Let G be a finite graph. Then the canonical semi-ring $R(G)$ has naturally a graded semi-ring structure over \mathbb{Z}^{trop} . For any divisor D on G , in general, the direct sum $\bigoplus_{m=0}^{\infty} R(G, mD)$ has naturally a graded semi-ring structure over \mathbb{Z}^{trop} .*

3.2. Proof of Theorem 1.2(b).

Theorem 3.6 (Theorem 1.2(b)). *For any positive integer n , there exists a finite graph G_n such that the canonical semi-ring $R(G_n)$ is not generated in degree at most $n - 1$.*

Proof. The idea of the proof is to construct a similar rational function as that in the proof of Lemma 2.8. So we omit the detail.

Let G be the finite graph with two vertices and three edges each of which connects the vertices. Let G_n be the finite graph obtained by replacing each edge of G with a segment which consists of $(2n - 1)$ edges. Note that G_n has $(6n - 4)$ vertices and $(6n - 3)$ edges. Let r be the $(n + 1)$ -th vertex counted from p on a segment, where p is a vertex of valence different from 2. The canonical divisor K_{G_n} is equal to that of G by definition, and the divisor nK_{G_n} is linearly equivalent to

$[p] + (2n - 1)[r]$. It follows that there exists an extremal f of $R(G_n, nK_{G_n})$ such that $nK_{G_n} + \text{div}(f)$ is equal to $[p] + (2n - 1)[r]$. Suppose that f is generated by the elements of $\bigoplus_{m=0}^{n-1} R(G_n, mK_{G_n})$ over \mathbb{Z}^{trop} . Then we may assume that f is equal to $g \odot h$, where g and h are the extremal of $R(G_n, lK_{G_n})$ and $R(G_n, kK_{G_n})$, respectively, and $l + k$ is equal to n . It follows that

$$\begin{aligned} lK_{G_n} + \text{div}(g) &= [p] + (2l - 1)[r], \\ kK_{G_n} + \text{div}(h) &= 2k[r]. \end{aligned}$$

Let u and w be the second vertex counted from p on each segment different from e , where e is the segment on which r is a vertex. After some calculations, we get

$$\begin{aligned} h(p) - h(q) &= k + (2n - 1)(h(u) + h(w) - 2h(p)), \\ h(p) - h(q) &= (2n - 1)(h(p) - h(u)). \end{aligned}$$

Then we have

$$k = (2n - 1)(3h(p) - 2h(u) - h(w)),$$

and $\frac{k}{2n-1}$ is an integer.

Hence $k \geq 2n - 1 \geq n$ and this contradicts $k \leq n - 1$. Therefore f is not generated by the elements of $\bigoplus_{m=0}^{n-1} R(G_n, mK_{G_n})$ over \mathbb{Z}^{trop} . \square

3.3. Proof of Theorem 1.2(a). We prove Theorem 1.2(a) in the following generalized form (where the canonical divisor K_G is replaced by any divisor D on G).

Theorem 3.7. *Let G be a finite graph and let D be a divisor on G . Then $\bigoplus_{m=0}^{\infty} R(G, mD)$ is finitely generated as a graded semi-ring over \mathbb{Z}^{trop} .*

Proof. Let the vertices $V(G) = \{v_1, \dots, v_n\}$, and let Δ be the graph Laplacian, that is, the $n \times n$ symmetric matrix such that each entry Δ_{ij} is equal to the number of edges which connect v_i to v_j if $i \neq j$, and the value $-\text{val}(v_i)$ if $i = j$. Using the Laplacian, we can describe $R(G, lD)$ as

$$R(G, lD) = \{f \in \text{Rat}(G) \mid \Delta \cdot {}^t(f(v_1), \dots, f(v_n)) + l \cdot {}^t(D(v_1), \dots, D(v_n)) \geq 0\}.$$

We identify $R(G, lD)$ with the lattice points of a polyhedron P_l in \mathbb{R}^n by a map $\Psi_l : R(G, lD) \rightarrow \mathbb{R}^n$ which maps f to ${}^t(f(v_1), \dots, f(v_n))$, where P_l is a polyhedron of the form

$$P_l = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \Delta \cdot {}^t(x_1, \dots, x_n) + l \cdot {}^t(D(v_1), \dots, D(v_n)) \geq 0\}.$$

By the fundamental theorem of polyhedra, it follows that P_l is a convex hull of finitely many vectors.

Let C be the cone obtained by coning P_1 , that is, the cone of the form

$$C = \{\lambda \cdot {}^t(u, 1) \in \mathbb{R}^{n+1} \mid u \in P_1, \lambda \in \mathbb{R}_{\geq 0}\}.$$

Note that C is a finitely generated cone.

Claim 3.8. *The lattice points of C whose $(n + 1)$ -th coordinate is equal to m can be identified with the elements of $R(G, mD)$.*

Proof. By the above identification, it is sufficient to show that each lattice point of C whose $(n+1)$ -th coordinate is equal to m corresponds to a lattice point of P_m . Let ${}^t(u, m) = m \cdot {}^t(\frac{1}{m} \cdot u, 1)$ be a lattice point of C . By definition, $\frac{1}{m} \cdot u$ is an element of P_1 . Since

$$\Delta \cdot {}^t(\frac{1}{m}u_1, \dots, \frac{1}{m}u_n) + {}^t(D(v_1), \dots, D(v_n)) \geq 0,$$

it follows that

$$\Delta \cdot {}^t(u_1, \dots, u_n) + m \cdot {}^t(D(v_1), \dots, D(v_n)) \geq 0.$$

Hence, u is a lattice point of P_m .

Conversely, let u be a lattice point of P_m and let $w \in \mathbb{R}^n$ be the vector whose i -th coordinate w_i is equal to $\frac{1}{m}u_i$. Then w is an element of P_1 in the same way as above. In particular, ${}^t(u, m) = m \cdot {}^t(w, 1)$ is a lattice point of C .

Thus each lattice point of C whose $(n+1)$ -th coordinate is equal to m corresponds to a lattice point of P_m . Therefore, the statement holds. \square

Moreover, the sum of the lattice points of C corresponds to the product of the elements of $R(G, mD)$ for some m . This follows from the same reason as the above correspondence, so we omit the detail.

By this correspondence and the Gordan's lemma (see [Fu, p.12, Proposition 1]), all the elements of $R(G, mD)$ for any m is generated by the elements of $R(G, nD)$ for finitely many n over \mathbb{Z}^{trop} .

Since the semi-ring $\bigoplus_{m=0}^{\infty} R(G, mD)$ is defined as a direct sum, all the elements of $\bigoplus_{m=0}^{\infty} R(G, mD)$ is generated by finitely many elements over \mathbb{Z}^{trop} . \square

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